

# **$p$ -ADIC DYNAMICAL SYSTEMS OF $(2, 2)$ -RATIONAL FUNCTIONS WITH UNIQUE FIXED POINT**

U.A. ROZIKOV, I.A. SATTAROV

**ABSTRACT.** We consider a family of  $(2, 2)$ -rational functions given on the set of complex  $p$ -adic field  $\mathbb{C}_p$ . Each such function has a unique fixed point. We study  $p$ -adic dynamical systems generated by the  $(2, 2)$ -rational functions. We show that the fixed point is indifferent and therefore the convergence of the trajectories is not the typical case for the dynamical systems. Siegel disks of these dynamical systems are found. We obtain an upper bound for the set of limit points of each trajectory, i.e., we determine a sufficiently small set containing the set of limit points. For each  $(2, 2)$ -rational function on  $\mathbb{C}_p$  there are two points  $\hat{x}_1 = \hat{x}_1(f)$ ,  $\hat{x}_2 = \hat{x}_2(f) \in \mathbb{C}_p$  which are zeros of its denominator. We give explicit formulas of radiuses of spheres (with the center at the fixed point) containing some points such that the trajectories (under actions of  $f$ ) of the points after a finite step come to  $\hat{x}_1$  or  $\hat{x}_2$ . Moreover for a class of  $(2, 2)$ -rational functions we study ergodicity properties of the dynamical systems on the set of  $p$ -adic numbers  $\mathbb{Q}_p$ . For each such function we describe all possible invariant spheres. We show that the  $p$ -adic dynamical system reduced on each invariant sphere is not ergodic with respect to Haar measure.

## 1. INTRODUCTION

We study dynamical systems generated by a rational function. A function is called a  $(n, m)$ -rational function if and only if it can be written in the form  $f(x) = \frac{P_n(x)}{Q_m(x)}$ , where  $P_n(x)$  and  $Q_m(x)$  are polynomial functions with degree  $n$  and  $m$  respectively,  $Q_m(x)$  is not the zero polynomial.

It is known that analytic functions play a fundamental role in complex analysis and rational functions play an analogous role in  $p$ -adic analysis [8], [20]. It is therefore natural to study dynamics generated by rational functions in  $p$ -adic analysis. In this paper we consider  $(2, 2)$ -rational functions on the field of complex  $p$ -adic numbers and study behavior of trajectories of the dynamical systems generated by such functions.

The  $p$ -adic dynamical systems arise in Diophantine geometry in the constructions of canonical heights, used for counting rational points on algebraic varieties over a number field, as in [7]. Moreover  $p$ -adic dynamical systems are effective in computer science (straight line programs), in numerical analysis and in simulations (pseudorandom numbers), uniform distribution of sequences, cryptography (stream ciphers,  $T$ -functions), combinatorics (Latin squares), automata theory and formal languages, genetics. The monograph [4] contains the corresponding survey. For newer results see [1], [5], [6]- [23].

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Let us briefly mention papers which are devoted to dynamical systems of  $(n, m)$ -rational functions (this is not a complete review of  $p$ -adic dynamical systems of rational functions). A polynomial function can be considered as a  $(n, 0)$ -rational function (see for example, [9]). Therefore, we start from review of such functions. The most studied discrete  $p$ -adic dynamical systems (iterations of maps) are the so-called monomial systems.

In [3], [12] the behavior of a  $p$ -adic dynamical system  $f(x) = x^n$  in the fields of  $p$ -adic numbers  $\mathbb{Q}_p$  and  $\mathbb{C}_p$  were studied.

In [2] the properties of the nonlinear  $p$ -adic dynamic system  $f(x) = x^2 + c$  with a single parameter  $c$  (i.e., a  $(2, 0)$ -rational function) on the integer  $p$ -adic numbers  $\mathbb{Z}_p$  are investigated. This dynamic system illustrates possible brain behaviors during remembering.

In [11], dynamical systems defined by the functions  $f_q(x) = x^n + q(x)$ , where the perturbation  $q(x)$  is a polynomial whose coefficients have small  $p$ -adic absolute value, was studied.

In [15], [18] the dynamical systems associated with the function  $f(x) = x^3 + ax^2$  on the set of  $p$ -adic numbers is studied. More general form of this function, i.e.,  $f(x) = x^{2n+1} + ax^{n+1}$ , is considered in [17].

Papers [10], [16] (see also references therein) are devoted to  $(1, 1)$ -rational  $p$ -adic dynamical systems.

In [1] and [13] the trajectories of an arbitrary  $(2, 1)$ -rational  $p$ -adic dynamical systems in a complex  $p$ -adic field  $\mathbb{C}_p$  are studied.

The paper [21] is devoted to a  $(3, 2)$ -rational  $p$ -adic dynamical system in  $\mathbb{C}_p$ , when there exists a unique fixed point.

In [22] we continued investigation of the  $(3, 2)$ -rational  $p$ -adic dynamical systems in  $\mathbb{C}_p$ , when there are two fixed points.

In this paper we investigate behavior of trajectory of a  $(2, 2)$ -rational  $p$ -adic dynamical system in  $\mathbb{C}_p$ .

The paper is organized as follows: in Section 2 we give some preliminaries. Section 3 contains the definition of the  $(2, 2)$ -rational function and main results about behavior of trajectories of the  $p$ -adic dynamical system. Siegel disks of these dynamical systems are studied. We obtain an upper bound for the set of limit points of each trajectory. We give explicit formulas of radiuses of spheres, with the center at the fixed point, containing some points such that the trajectories of the points after a finite step come to zeros of the denominator of the rational function. In Section 4 for a class of  $(2, 2)$ -rational functions we study ergodicity properties of the dynamical systems on the set of  $p$ -adic numbers  $\mathbb{Q}_p$ . For each such function we describe all possible invariant spheres. We study ergodicity of each  $p$ -adic dynamical system with respect to Haar measure reduced on each invariant sphere. It is proved that the dynamical systems are not ergodic.

## 2. PRELIMINARIES

**2.1.  $p$ -adic numbers.** Let  $\mathbb{Q}$  be the field of rational numbers. The greatest common divisor of the positive integers  $n$  and  $m$  is denoted by  $(n, m)$ . Every rational number  $x \neq 0$  can be represented in the form  $x = p^r \frac{n}{m}$ , where  $r, n \in \mathbb{Z}$ ,  $m$  is a positive integer,  $(p, n) = 1$ ,  $(p, m) = 1$  and  $p$  is a fixed prime number.

The  $p$ -adic norm of  $x$  is given by

$$|x|_p = \begin{cases} p^{-r}, & \text{for } x \neq 0, \\ 0, & \text{for } x = 0. \end{cases}$$

It has the following properties:

- 1)  $|x|_p \geq 0$  and  $|x|_p = 0$  if and only if  $x = 0$ ,
- 2)  $|xy|_p = |x|_p |y|_p$ ,
- 3) the strong triangle inequality

$$|x + y|_p \leq \max\{|x|_p, |y|_p\},$$

3.1) if  $|x|_p \neq |y|_p$  then  $|x + y|_p = \max\{|x|_p, |y|_p\}$ ,

3.2) if  $|x|_p = |y|_p$  then  $|x + y|_p \leq |x|_p$ ,

this is a non-Archimedean one.

The completion of  $\mathbb{Q}$  with respect to  $p$ -adic norm defines the  $p$ -adic field which is denoted by  $\mathbb{Q}_p$  (see [14]).

The algebraic completion of  $\mathbb{Q}_p$  is denoted by  $\mathbb{C}_p$  and it is called *complex  $p$ -adic numbers*. For any  $a \in \mathbb{C}_p$  and  $r > 0$  denote

$$U_r(a) = \{x \in \mathbb{C}_p : |x - a|_p < r\}, \quad V_r(a) = \{x \in \mathbb{C}_p : |x - a|_p \leq r\},$$

$$S_r(a) = \{x \in \mathbb{C}_p : |x - a|_p = r\}.$$

A function  $f : U_r(a) \rightarrow \mathbb{C}_p$  is said to be *analytic* if it can be represented by

$$f(x) = \sum_{n=0}^{\infty} f_n(x - a)^n, \quad f_n \in \mathbb{C}_p,$$

which converges uniformly on the ball  $U_r(a)$ .

**2.2. Dynamical systems in  $\mathbb{C}_p$ .** Recall some known facts concerning dynamical systems  $(f, U)$  in  $\mathbb{C}_p$ , where  $f : x \in U \rightarrow f(x) \in U$  is an analytic function and  $U = U_r(a)$  or  $\mathbb{C}_p$  (see for example [19]).

Now let  $f : U \rightarrow U$  be an analytic function. Denote  $f^n(x) = \underbrace{f \circ \dots \circ f}_n(x)$ .

If  $f(x_0) = x_0$  then  $x_0$  is called a *fixed point*. The set of all fixed points of  $f$  is denoted by  $\text{Fix}(f)$ . A fixed point  $x_0$  is called an *attractor* if there exists a neighborhood  $U(x_0)$  of  $x_0$  such that for all points  $x \in U(x_0)$  it holds  $\lim_{n \rightarrow \infty} f^n(x) = x_0$ . If  $x_0$  is an attractor then its *basin of attraction* is

$$A(x_0) = \{x \in \mathbb{C}_p : f^n(x) \rightarrow x_0, n \rightarrow \infty\}.$$

A fixed point  $x_0$  is called *repeller* if there exists a neighborhood  $U(x_0)$  of  $x_0$  such that  $|f(x) - x_0|_p > |x - x_0|_p$  for  $x \in U(x_0)$ ,  $x \neq x_0$ .

Let  $x_0$  be a fixed point of a function  $f(x)$ . Put  $\lambda = f'(x_0)$ . The point  $x_0$  is attractive if  $0 < |\lambda|_p < 1$ , *indifferent* if  $|\lambda|_p = 1$ , and repelling if  $|\lambda|_p > 1$ .

The ball  $U_r(x_0)$  (contained in  $V$ ) is said to be a *Siegel disk* if each sphere  $S_\rho(x_0)$ ,  $\rho < r$  is an invariant sphere of  $f(x)$ , i.e. if  $x \in S_\rho(x_0)$  then all iterated points  $f^n(x) \in S_\rho(x_0)$  for

all  $n = 1, 2, \dots$ . The union of all Siegel disks with the center at  $x_0$  is said to a *maximum Siegel disk* and is denoted by  $SI(x_0)$ .

### 3. (2, 2)-RATIONAL $p$ -ADIC DYNAMICAL SYSTEMS

In this paper we consider the dynamical system associated with the (2, 2)-rational function  $f : \mathbb{C}_p \rightarrow \mathbb{C}_p$  defined by

$$f(x) = \frac{ax^2 + bx + c}{x^2 + dx + e}, \quad a \neq 0, \quad |b - ad|_p + |c - ae|_p \neq 0, \quad a, b, c, d, e \in \mathbb{C}_p. \quad (3.1)$$

where  $x \neq \hat{x}_{1,2} = \frac{-d \pm \sqrt{d^2 - 4e}}{2}$ .

**Remark 1.** We note that if  $b = ad$  and  $c = ae$  then from (3.1) we get  $f(x) = a$ , i.e.,  $f$  becomes a constant function. Therefore we assumed  $b \neq ad$  or  $c \neq ae$ .

It is easy to see that for (2, 2)-rational function (3.1) the equation  $f(x) = x$  for fixed points is equivalent to the equation

$$x^3 + (d - a)x^2 + (e - b)x - c = 0. \quad (3.2)$$

Since  $\mathbb{C}_p$  is algebraic close the equation (3.2) may have three solutions with one of the following relations:

- (i). One solution having multiplicity three;
- (ii). Two solutions, one of which has multiplicity two;
- (iii). Three distinct solutions.

In this paper we investigate the behavior of trajectories of an arbitrary (2, 2)-rational dynamical system in complex  $p$ -adic field  $\mathbb{C}_p$  when there is unique fixed point for  $f$ , i.e., we consider the case (i).

The following lemma gives a criterion on parameters of the function (3.1) guaranteeing the uniqueness of its fixed point.

**Lemma 1.** *The function (3.1) has unique fixed point if and only if*

$$\frac{a - d}{3} = -\sqrt{\frac{e - b}{3}} = \sqrt[3]{c} \quad \text{or} \quad \frac{a - d}{3} = \sqrt{\frac{e - b}{3}} = \sqrt[3]{c}. \quad (3.3)$$

*Proof. Necessariness.* Assume (3.1) has a unique fixed point, say  $x_0$ . Then the LHS of equation (3.2) (which is equivalent to  $f(x) = x$ ) can be written as

$$x^3 + (d - a)x^2 + (e - b)x - c = (x - x_0)^3.$$

Consequently,

$$\begin{cases} 3x_0 = a - d \\ 3x_0^2 = e - b \\ x_0^3 = c \end{cases},$$

which gives

$$x_0 = \frac{a - d}{3} = \pm \sqrt{\frac{e - b}{3}} = \sqrt[3]{c}.$$

*Sufficiency.* Assume the coefficients of (3.1) satisfy (3.3). Then it can be written as

$$f(x) = \frac{ax^2 + bx + \left(\frac{a-d}{3}\right)^3}{x^2 + dx + \frac{(a-d)^2}{3} + b}, \quad a \neq 0, \quad b \neq -2a^2 \quad \text{or} \quad d \neq -2a, \quad a, b, d \in \mathbb{C}_p. \quad (3.4)$$

In this case the equation  $f(x) = x$  can be written as

$$\left(x - \frac{a-d}{3}\right)^3 = 0.$$

Thus  $f(x)$  has unique fixed point  $x_0 = \frac{a-d}{3}$ . □

It follows from this lemma that if the function (3.1) has unique fixed point then it has the form (3.4). Thus we study the dynamical system  $(f, \mathbb{C}_p)$  with  $f$  given by (3.4).

For (3.4) we have

$$f'(x_0) = f'\left(\frac{a-d}{3}\right) = 1,$$

i.e., the point  $x_0$  is an indifferent point for (3.4).

In (3.4) one assumes  $x^2 + dx + \frac{(a-d)^2}{3} + b \neq 0$ , i.e.,  $x \neq x_{1,2} = -\frac{d}{2} \pm \sqrt{\frac{d^2}{4} - \frac{(a-d)^2}{3} - b}$ .

For any  $x \in \mathbb{C}_p$ ,  $x \neq x_{1,2}$ , by simple calculations we get

$$|f(x) - x_0|_p = |x - x_0|_p \cdot \frac{\left|\frac{2a+d}{3}(x - x_0) + (x_0 - x_1)(x_0 - x_2)\right|_p}{|(x - x_0) + (x_0 - x_1)|_p |(x - x_0) + (x_0 - x_2)|_p}. \quad (3.5)$$

Denote

$$\mathcal{P} = \{x \in \mathbb{C}_p : \exists n \in \mathbb{N} \cup \{0\}, f^n(x) \in \{x_1, x_2\}\},$$

$$\delta = \left|\frac{2a+d}{3}\right|_p, \quad \alpha = |x_0 - x_1|_p \quad \text{and} \quad \beta = |x_0 - x_2|_p.$$

**Lemma 2.** 1. If  $\left|\sqrt{\frac{d^2}{4} - \frac{(a-d)^2}{3} - b}\right|_p \neq \left|\frac{2a+d}{6}\right|_p$ , then  $\alpha = \beta$ .

2. If  $\left|\sqrt{\frac{d^2}{4} - \frac{(a-d)^2}{3} - b}\right|_p = \left|\frac{2a+d}{6}\right|_p$ , then

- $\alpha \leq \delta$  and  $\beta \leq \delta$  for all  $p \geq 3$ .
- $\alpha \leq 2\delta$  and  $\beta \leq 2\delta$  for  $p = 2$ .

*Proof.* This follows from properties of the norm  $|\cdot|_p$ . □

**Remark 2.** It is easy to see that  $x_0 - x_1$  and  $x_0 - x_2$  are symmetric in (3.5), i.e., if we replace them then RHS of (3.5) does not change. Therefore we consider the dynamical system  $(f, \mathbb{C}_p \setminus \mathcal{P})$  for cases  $\alpha = \beta$  and  $\alpha < \beta$ .

3.1. **Case:**  $\alpha = \beta$ . Let us consider the following functions:

For  $\alpha > \delta$  define the function  $\varphi_{\alpha,\delta} : [0, +\infty) \rightarrow [0, +\infty)$  by

$$\varphi_{\alpha,\delta}(r) = \begin{cases} r, & \text{if } r < \alpha \\ \alpha^*, & \text{if } r = \alpha \\ \frac{\alpha^2}{r}, & \text{if } \alpha < r < \frac{\alpha^2}{\delta} \\ \delta^*, & \text{if } r = \frac{\alpha^2}{\delta} \\ \delta, & \text{if } r > \frac{\alpha^2}{\delta} \end{cases}$$

where  $\alpha^*$  and  $\delta^*$  some positive numbers with  $\alpha^* \geq \alpha$ ,  $\delta^* \leq \delta$ .

For  $\alpha < \delta$  define the function  $\phi_{\alpha,\delta} : [0, +\infty) \rightarrow [0, +\infty)$  by

$$\phi_{\alpha,\delta}(r) = \begin{cases} r, & \text{if } r < \frac{\alpha^2}{\delta} \\ \alpha', & \text{if } r = \frac{\alpha^2}{\delta} \\ \frac{\delta r^2}{\alpha^2}, & \text{if } \frac{\alpha^2}{\delta} < r < \alpha \\ \delta', & \text{if } r = \alpha \\ \delta, & \text{if } r > \alpha \end{cases}$$

where  $\alpha'$  and  $\delta'$  some positive numbers with  $\alpha' \leq \frac{\alpha^2}{\delta}$ ,  $\delta' \geq \delta$ .

For  $\alpha = \delta$  define the function  $\psi_\alpha : [0, +\infty) \rightarrow [0, +\infty)$  by

$$\psi_\alpha(r) = \begin{cases} r, & \text{if } r < \alpha \\ \hat{\alpha}, & \text{if } r = \alpha \\ \alpha, & \text{if } r > \alpha \end{cases}$$

where  $\hat{\alpha}$  some positive number.

Using the formula (3.5) we easily get the following:

**Lemma 3.** *If  $\alpha = \beta$  and  $x \in S_r(x_0)$ , then the following formula holds for function (3.4)*

$$|f^n(x) - x_0|_p = \begin{cases} \varphi_{\alpha,\delta}^n(r), & \text{if } \alpha > \delta \\ \phi_{\alpha,\delta}^n(r), & \text{if } \alpha < \delta \\ \psi_\alpha^n(r), & \text{if } \alpha = \delta. \end{cases}$$

Thus the  $p$ -adic dynamical system  $f^n(x), n \geq 1, x \in \mathbb{C}_p \setminus \mathcal{P}$  is related to the real dynamical systems generated by  $\varphi_{\alpha,\delta}$ ,  $\phi_{\alpha,\delta}$  and  $\psi_\alpha$ . Now we are going to study these (real) dynamical systems.

**Lemma 4.** *If  $\alpha > \delta$ , then the dynamical system generated by  $\varphi_{\alpha,\delta}(r)$  has the following properties:*

1.  $\text{Fix}(\varphi_{\alpha,\delta}) = \{r : 0 \leq r < \alpha\} \cup \{\alpha : \text{if } \alpha^* = \alpha\}$ .

2. If  $r > \alpha$ , then

$$\varphi_{\alpha,\delta}^n(r) = \begin{cases} \frac{\alpha^2}{r}, & \text{for all } \alpha < r < \frac{\alpha^2}{\delta} \\ \delta^*, & \text{for } r = \frac{\alpha^2}{\delta} \\ \delta, & \text{for all } r > \frac{\alpha^2}{\delta} \end{cases}$$

for any  $n \geq 1$ .

3. If  $r = \alpha$  and  $\alpha^* > \alpha$ , then

$$\varphi_{\alpha,\delta}^n(r) = \begin{cases} \frac{\alpha^2}{\alpha^*}, & \text{if } \alpha < \alpha^* < \frac{\alpha^2}{\delta} \\ \delta^*, & \text{if } \alpha^* = \frac{\alpha^2}{\delta} \\ \delta, & \text{if } \alpha^* > \frac{\alpha^2}{\delta} \end{cases}$$

for any  $n \geq 2$ .

*Proof.* 1. This is the result of a simple analysis of the equation  $\varphi_{\alpha,\delta}(r) = r$ .

2. If  $r > \delta$ , then

$$\varphi_{\alpha,\delta}(r) = \begin{cases} \frac{\alpha^2}{r}, & \text{if } \alpha < r < \frac{\alpha^2}{\delta} \\ \delta^*, & \text{if } r = \frac{\alpha^2}{\delta} \\ \delta, & \text{if } r > \frac{\alpha^2}{\delta}. \end{cases}$$

Consequently,

$$\alpha < r < \frac{\alpha^2}{\delta} \Rightarrow \delta < \frac{\alpha^2}{r} < \alpha \Rightarrow \varphi_{\alpha,\delta}(r) < \alpha.$$

If  $r \geq \frac{\alpha^2}{\delta}$ , then by  $\delta^* \leq \delta < \alpha$  we have  $\varphi_{\alpha,\delta}(r) < \alpha$ . Thus  $\varphi_{\alpha,\delta}(\varphi_{\alpha,\delta}(r)) = \varphi_{\alpha,\delta}(r)$ , i.e.,  $\varphi_{\alpha,\delta}(r)$  is a fixed point of  $\varphi_{\alpha,\delta}$  for any  $r > \alpha$ . Consequently, for each  $n \geq 1$  we have

$$\varphi_{\alpha,\delta}^n(r) = \begin{cases} \frac{\alpha^2}{r}, & \text{if } \alpha < r < \frac{\alpha^2}{\delta} \\ \delta^*, & \text{if } r = \frac{\alpha^2}{\delta} \\ \delta, & \text{if } r > \frac{\alpha^2}{\delta}. \end{cases}$$

3. The part 3 easily follows from the parts 1 and 2. □

**Lemma 5.** If  $\alpha < \delta$ , then the dynamical system generated by  $\phi_{\alpha,\delta}(r)$  has the following properties:

- A.  $\text{Fix}(\phi_{\alpha,\delta}) = \{r : 0 \leq r < \frac{\alpha^2}{\delta}\} \cup \{\frac{\alpha^2}{\delta} : \text{if } \alpha' = \frac{\alpha^2}{\delta}\} \cup \{\delta\}$ .
- B. If  $r > \frac{\alpha^2}{\delta}$ , then

$$\lim_{n \rightarrow \infty} \phi_{\alpha,\delta}^n(r) = \delta.$$

- C. If  $r = \frac{\alpha^2}{\delta}$  and  $\alpha' < \frac{\alpha^2}{\delta}$ , then  $\phi_{\alpha,\delta}^n(r) = \alpha'$  for all  $n \geq 1$ .

*Proof.* A. This is the result of a simple analysis of the equation  $\phi_{\alpha,\delta}(r) = r$ .

B. By definition of  $\phi_{\alpha,\delta}(r)$ , for  $r > \alpha$  we have  $\phi_{\alpha,\delta}(r) = \delta$ , i.e., the function is constant. Therefore

$$\lim_{n \rightarrow \infty} \phi_{\alpha,\delta}^n(r) = \delta.$$

For  $r = \alpha$  we have  $\phi_{\alpha,\delta}(\alpha) = \delta' \geq \delta$  and by condition  $\delta > \alpha$ , we get  $\phi_{\alpha,\delta}(\alpha) > \alpha$ . Consequently,

$$\lim_{n \rightarrow \infty} \phi_{\alpha,\delta}^n(\alpha) = \delta.$$

Assume now  $\frac{\alpha^2}{\delta} < r < \alpha$  then  $\phi_{\alpha,\delta}(r) = \frac{\delta r^2}{\alpha^2}$ ,  $\phi'_{\alpha,\delta}(r) = \frac{2\delta r}{\alpha^2} > 2$  and

$$\phi_{\alpha,\delta}\left(\left(\frac{\alpha^2}{\delta}, \alpha\right)\right) = \left(\frac{\alpha^2}{\delta}, \delta\right) \cup \{\delta'\}.$$

Since  $\phi'_{\alpha,\delta}(r) > 2$  for  $r \in (\frac{\alpha^2}{\delta}, \alpha)$  there exists  $n_0 \in \mathbb{N}$  such that  $\phi_{\alpha,\delta}^{n_0}(r) \in (\alpha, \delta)$ . Hence for  $n \geq n_0$  we get  $\phi_{\alpha,\delta}^n(r) > \alpha$  and consequently

$$\lim_{n \rightarrow \infty} \phi_{\alpha,\delta}^n(r) = \delta.$$

C. If  $r = \frac{\alpha^2}{\delta}$  and  $\alpha' < \frac{\alpha^2}{\delta}$  then  $\phi_{\alpha,\delta}(r) = \alpha' < \frac{\alpha^2}{\delta}$ . Moreover,  $\alpha'$  is a fixed point for the function  $\phi_{\alpha,\delta}$ . Thus for  $n \geq 1$  we obtain  $\phi_{\alpha,\delta}^n(r) = \alpha'$ .  $\square$

**Lemma 6.** *If  $\alpha = \delta$ , then the dynamical system generated by  $\psi_\alpha(r)$  has the following properties:*

- I.  $\text{Fix}(\psi_\alpha) = \{r : 0 \leq r < \alpha\} \cup \{\alpha : \text{if } \hat{\alpha} = \alpha\}$ .
- II. *If  $r > \alpha$ , then  $\psi_\alpha(r) = \alpha$ .*
- III. *Let  $r = \alpha$ .*
  - III.i) *If  $\hat{\alpha} < \alpha$ , then  $\psi_\alpha^n(r) = \hat{\alpha}$ , for any  $n \geq 1$ .*
  - III.ii) *If  $\hat{\alpha} > \alpha$ , then  $\psi_\alpha^2(\alpha) = \alpha$ .*

*Proof.* I. This is the result of a simple analysis of the equation  $\psi_\alpha(r) = r$ .

II. By definition of  $\psi_\alpha(r)$ , for any  $r > \alpha$  we have  $\psi_\alpha(r) = \alpha$ .

III. If  $r = \alpha$  then  $\psi_\alpha(r) = \hat{\alpha}$ .

For  $\hat{\alpha} \leq \alpha$  we have  $\psi_\alpha(\hat{\alpha}) = \hat{\alpha}$ . Thus for all  $n \geq 1$  one has  $\psi_\alpha^n(r) = \hat{\alpha}$ .

In case  $\hat{\alpha} > \alpha$  we have  $\psi_\alpha(\hat{\alpha}) = \alpha$ ,  $\psi_\alpha(\alpha) = \hat{\alpha}$ . Hence  $\psi_\alpha^2(\alpha) = \alpha$ .  $\square$

Now we shall apply these lemmas to study of the  $p$ -adic dynamical system generated by function (3.4).

For  $\alpha > \delta$  denote the following

$$\alpha^*(x) = |f(x) - x_0|_p, \quad \text{if } x \in S_\alpha(x_0)$$

and

$$\delta^*(x) = |f(x) - x_0|_p, \quad \text{if } x \in S_{\frac{\alpha^2}{\delta}}(x_0).$$

Then using Lemma 3 and Lemma 4 we obtain the following

**Theorem 1.** *If  $\alpha > \delta$ , then the  $p$ -adic dynamical system generated by function (3.4) has the following properties:*



1. 1.1)  $SI(x_0) = U_\alpha(x_0)$ .
- 1.2)  $\mathcal{P} \subset S_\alpha(x_0)$ .
2. If  $r > \alpha$  and  $x \in S_r(x_0)$ , then

$$f^n(x) \in \begin{cases} S_{\frac{\alpha^2}{r}}(x_0), & \text{for all } \alpha < r < \frac{\alpha^2}{\delta} \\ S_{\delta^*(x)}(x_0), & \text{for } r = \frac{\alpha^2}{\delta} \\ S_\delta(x_0), & \text{for all } r > \frac{\alpha^2}{\delta}, \end{cases}$$

for any  $n \geq 1$ .

3. If  $x \in S_\alpha(x_0) \setminus \mathcal{P}$ , then one of the following two possibilities holds:
  - 3.1) There exists  $k \in \mathbb{N}$  and  $\mu_k > \alpha$  such that  $f^k(x) \in S_{\mu_k}(x_0)$  and

$$f^m(x) \in \begin{cases} S_{\frac{\alpha^2}{\mu_k}}(x_0), & \text{for all } \alpha < \mu_k < \frac{\alpha^2}{\delta} \\ S_{\delta^*(f^k(x))}(x_0), & \text{for } \mu_k = \frac{\alpha^2}{\delta} \\ S_\delta(x_0), & \text{for all } \mu_k > \frac{\alpha^2}{\delta} \end{cases}$$

for any  $m \geq k+1$  and  $f^m(x) \in S_\alpha(x_0)$  if  $m \leq k-1$ .

- 3.2) The trajectory  $\{f^k(x), k \geq 1\}$  is a subset of  $S_\alpha(x_0)$ .

*Proof.* The part 2 easily follows from Lemma 3 and the part 2 of Lemma 4.

3. Take  $x \in S_\alpha(x_0) \setminus \mathcal{P}$  then we have

$$|f(x) - x_0|_p = \frac{\alpha^3}{|(x - x_0) + (x_0 - x_1)|_p |(x - x_0) + (x_0 - x_2)|_p} \geq \alpha.$$

If  $|f(x) - x_0|_p > \alpha$  then there is  $\mu_1 > \alpha$  such that  $f(x) \in S_{\mu_1}(x_0)$  and by part 2 we have

$$f^m(x) \in \begin{cases} S_{\frac{\alpha^2}{\mu_1}}(x_0), & \text{for all } \alpha < \mu_1 < \frac{\alpha^2}{\delta} \\ S_{\delta^*(f(x))}(x_0), & \text{for } \mu_1 = \frac{\alpha^2}{\delta} \\ S_\delta(x_0), & \text{for all } \mu_1 > \frac{\alpha^2}{\delta} \end{cases}$$

for any  $m \geq 2$ . So in this case  $k = 1$ .

If  $|f(x) - x_0|_p = \alpha$  then we consider the following

$$|f^2(x) - x_0|_p = \frac{\alpha^3}{|(f(x) - x_0) + (x_0 - x_1)|_p |(f(x) - x_0) + (x_0 - x_2)|_p} \geq \alpha.$$

Now, if  $|f^2(x) - x_0|_p > \alpha$  then there is  $\mu_2 > \alpha$  such that  $f^2(x) \in S_{\mu_2}(x_0)$  and by part 2 we get

$$f^m(x) \in \begin{cases} S_{\frac{\alpha^2}{\mu_2}}(x_0), & \text{for all } \alpha < \mu_2 < \frac{\alpha^2}{\delta} \\ S_{\delta^*(f^2(x))}(x_0), & \text{for } \mu_2 = \frac{\alpha^2}{\delta} \\ S_\delta(x_0), & \text{for all } \mu_2 > \frac{\alpha^2}{\delta} \end{cases}$$

for any  $m \geq 3$ . So in this case  $k = 2$ .

If  $|f^2(x) - x_0|_p = \alpha$  then we can continue the argument and get the following inequality

$$|f^k(x) - x_0|_p \geq \alpha.$$

Hence in each step we may have two possibilities:  $|f^k(x) - x_0|_p = \alpha$  or  $|f^k(x) - x_0|_p > \alpha$ . In case  $|f^k(x) - x_0|_p > \alpha$  there exists  $\mu_k$  such that  $f^k(x) \in S_{\mu_k}(x_0)$ , and

$$f^m(x) \in \begin{cases} S_{\frac{\alpha^2}{\mu_k}}(x_0), & \text{for all } \alpha < \mu_k < \frac{\alpha^2}{\delta} \\ S_{\delta^*(f^k(x))}(x_0), & \text{for } \mu_k = \frac{\alpha^2}{\delta} \\ S_{\delta}(x_0), & \text{for all } \mu_k > \frac{\alpha^2}{\delta} \end{cases}$$

for any  $m \geq k + 1$ . If  $|f^k(x) - x_0|_p = \alpha$  for any  $k \in \mathbb{N}$  then  $\{f^k(x), k \geq 1\} \subset S_{\alpha}(x_0)$ .

1. By parts 2 and 3 of theorem we know that  $S_r(x_0)$  is not an invariant of  $f$  for  $r \geq \alpha$ . Consequently,  $SI(x_0) \subset U_{\alpha}(x_0)$ .

By Lemma 3 and part 1 of Lemma 4 if  $r < \alpha$  and  $x \in S_r(x_0)$  then  $|f^n(x) - x_0|_p = \varphi_{\alpha, \delta}^n(r) = r$ , i.e.,  $f^n(x) \in S_r(x_0)$ . Hence  $U_{\alpha}(x_0) \subset SI(x_0)$  and thus  $SI(x_0) = U_{\alpha}(x_0)$ .

Since  $|x_0 - x_1|_p = |x_0 - x_2|_p = \alpha$  we have  $x_i \notin U_{\alpha}(x_0)$ ,  $i = 1, 2$ . From  $f(U_{\alpha}(x_0)) \subset U_{\alpha}(x_0)$  it follows that

$$U_{\alpha}(x_0) \cap \mathcal{P} = \{x \in U_{\alpha}(x_0) : \exists n \in \mathbb{N} \cup \{0\}, f^n(x) \in \{x_1, x_2\}\} = \emptyset.$$

By part 2 of theorem for  $r > \alpha$  we have  $f(S_r(x_0)) \subset U_{\alpha}(x_0)$ . Thus

$$(\mathbb{C}_p \setminus V_{\alpha}(x_0)) \cap \mathcal{P} = \emptyset,$$

i.e.,  $\mathcal{P} \subset S_{\alpha}(x_0)$ . □

By Lemma 3 and Lemma 5 we get

**Theorem 2.** *If  $\alpha < \delta$ , then the  $p$ -adic dynamical system generated by function (3.4) has the following properties:*

A. A.a)  $SI(x_0) = U_{\frac{\alpha^2}{\delta}}(x_0)$ .

A.b)  $f(S_{\delta}(x_0)) \subset S_{\delta}(x_0)$ , i.e.,  $S_{\delta}(x_0)$  is an invariant.

B. If  $r > \frac{\alpha^2}{\delta}$  and  $x \in S_r(x_0) \setminus \mathcal{P}$ , then

$$\lim_{n \rightarrow \infty} f^n(x) \in S_{\delta}(x_0).$$

C. If  $r = \frac{\alpha^2}{\delta}$ , then one of the following two possibilities holds:

C.a) There exists  $k \in \mathbb{N}$  and  $\mu_k < \frac{\alpha^2}{\delta}$  such that  $f^m(x) \in S_{\mu_k}(x_0)$  for any  $m \geq k$  and  $f^m(x) \in S_{\frac{\alpha^2}{\delta}}(x_0)$  if  $m \leq k - 1$ .

C.b) The trajectory  $\{f^k(x), k \geq 1\}$  is a subset of  $S_{\frac{\alpha^2}{\delta}}(x_0)$ .

*Proof.* A. By Lemma 3 and part A of Lemma 5 we see that spheres  $S_r(x_0)$  and  $S_{\delta}(x_0)$  are invariant for  $f$  for any  $r < \frac{\alpha^2}{\delta}$ . Thus  $SI(x_0) = U_{\frac{\alpha^2}{\delta}}(x_0)$ .

B. Follows from Lemma 3 and part B of Lemma 5.

C. If  $x \in S_{\frac{\alpha^2}{\delta}}(x_0)$  then we have

$$|f(x) - x_0|_p = \frac{\left| \frac{2a+d}{3}(x - x_0) + (x_0 - x_1)(x_0 - x_2) \right|_p}{\delta} \leq \frac{\alpha^2}{\delta}.$$

If  $|f(x) - x_0|_p < \frac{\alpha^2}{\delta}$  then there is  $\mu_1 < \frac{\alpha^2}{\delta}$  such that  $f^m(x) \in S_{\mu_1}(x_0)$  for any  $m \geq 1$  (see part A of Lemma 5). So in this case  $k = 1$ .

If  $|f(x) - x_0|_p = \frac{\alpha^2}{\delta}$  then we consider the following

$$|f^2(x) - x_0|_p = \frac{\left| \frac{2a+d}{3}(f(x) - x_0) + (x_0 - x_1)(x_0 - x_2) \right|_p}{\delta} \leq \frac{\alpha^2}{\delta}.$$

Now, if  $|f^2(x) - x_0|_p < \frac{\alpha^2}{\delta}$  then there is  $\mu_2 < \frac{\alpha^2}{\delta}$  such that  $f^m(x) \in S_{\mu_2}(x_0)$  for any  $m \geq 2$ . So in this case  $k = 2$ .

If  $|f^2(x) - x_0|_p = \frac{\alpha^2}{\delta}$  then we can continue the argument and get the following inequality

$$|f^k(x) - x_0|_p \leq \frac{\alpha^2}{\delta}.$$

Hence in each step we may have two possibilities:  $|f^k(x) - x_0|_p = \frac{\alpha^2}{\delta}$  or  $|f^k(x) - x_0|_p < \frac{\alpha^2}{\delta}$ . In case  $|f^k(x) - x_0|_p < \frac{\alpha^2}{\delta}$  there exists  $\mu_k$  such that  $f^m(x) \in S_{\mu_k}(x_0)$  for any  $m \geq k$ . If  $|f^k(x) - x_0|_p = \frac{\alpha^2}{\delta}$  for any  $k \in \mathbb{N}$  then  $\{f^k(x), k \geq 1\} \subset S_{\frac{\alpha^2}{\delta}}(x_0)$ .  $\square$

We note that  $\mathcal{P}$  has the following form

$$\mathcal{P} = \bigcup_{k=0}^{\infty} \mathcal{P}_k, \quad \mathcal{P}_k = \{x \in \mathbb{C}_p : f^k(x) \in \{x_1, x_2\}\}.$$

**Theorem 3.** *If  $\alpha < \delta$ , then*

1.  $\mathcal{P}_k \neq \emptyset$ , for any  $k = 0, 1, 2, \dots$ .
2.  $\mathcal{P}_k \subset S_{r_k}(x_0)$ , where  $r_k = \alpha \cdot \left(\frac{\alpha}{\delta}\right)^{\frac{2^k-1}{2^k}}$ ,  $k = 0, 1, 2, \dots$ .

*Proof.* 1. In case  $k = 0$  we have  $\mathcal{P}_0 = \{x_1, x_2\} \neq \emptyset$ .

Assume for  $k = n$  that  $\mathcal{P}_n = \{x \in \mathbb{C}_p : f^n(x) \in \{x_1, x_2\}\} \neq \emptyset$ .

Now for  $k = n + 1$  to prove  $\mathcal{P}_{n+1} = \{x \in \mathbb{C}_p : f^{n+1}(x) \in \{x_1, x_2\}\} \neq \emptyset$  we have to show that the following equation has at least one solution:

$$f^{n+1}(x) = x_i, \quad \text{for some } i = 1, 2.$$

By our assumption  $\mathcal{P}_n \neq \emptyset$  there exists  $y \in \mathcal{P}_n$  such that  $f^n(y) \in \{x_1, x_2\}$ . Now we show that there exists  $x$  such that  $f(x) = y$ . We note that the equation  $f(x) = y$  can be written as

$$(a - y)x^2 + (b - dy)x + \left(\frac{a - d}{3}\right)^3 - y \left[\frac{(a - d)^2}{3} + b\right] = 0. \quad (3.6)$$

We have  $|a - x_0|_p = \left|\frac{2a+d}{3}\right|_p = \delta$ , consequently,  $a \in S_{\delta}(x_0)$ . Since  $x_1, x_2 \in S_{\alpha}(x_0)$  and by the part A.b) of Theorem 2 we know that  $S_{\delta}(x_0)$  is an invariant we get  $\mathcal{P} \cap S_{\delta}(x_0) = \emptyset$ , for

$\alpha < \delta$ . Thus  $a \notin \mathcal{P}$ , consequently,  $a - y \neq 0$ . Since  $\mathbb{C}_p$  is algebraic closed the equation (3.6) has two solutions, say  $x = t_1, t_2$ . For  $x \in \{t_1, t_2\}$  we get

$$f^{n+1}(x) = f^n(f(x)) = f^n(y) \in \{x_1, x_2\}.$$

Hence  $\mathcal{P}_{n+1} \neq \emptyset$ . Therefore, by induction we get

$$\mathcal{P}_k \neq \emptyset, \text{ for any } k = 0, 1, 2, \dots$$

2. We know  $|x_0 - x_1|_p = |x_0 - x_2|_p = \alpha$ . By condition  $\alpha < \delta$ , we get  $\alpha > \frac{\alpha^2}{\delta}$ . By (3.5) and part B of Lemma 5 for  $x \in S_\alpha(x_0)$ ,  $x \neq x_{1,2}$  we have

$$\lim_{n \rightarrow \infty} f^n(x) \in S_\delta(x_0),$$

i.e.,  $S_\alpha(x_0) \cap \mathcal{P} = \{x_1, x_2\} = \mathcal{P}_0$ . Denoting  $r_0 = \alpha$  we write  $\mathcal{P}_0 \subset S_{r_0}(x_0)$ . Now to find spheres containing the solutions of the equations

$$f^k(x) = x_i, \quad k = 1, 2, 3, \dots, \quad i = 1, 2.$$

We write the last equations in the form

$$f^k(x) - x_0 = x_i - x_0, \quad k = 1, 2, 3, \dots, \quad i = 1, 2.$$

For each  $k$  we want to find some  $r_k$  such that the solution  $x$  of  $f^k(x) = x_i$ , (for some  $i = 1, 2$ ) belongs to  $S_{r_k}(x_0)$ , i.e.,  $x \in S_{r_k}(x_0)$ . By Lemma 3 we should have

$$\phi_{\alpha, \delta}^k(r_k) = \alpha.$$

Now if we show that the last equation has unique solution  $r_k$  for each  $k$ , then we get

$$\mathcal{P}_k = \{x \in \mathbb{C}_p : f^k(x) = x_i, i = 1, 2\} \subset S_{r_k}(x_0).$$

By parts A and C of Lemma 5 we have  $\frac{\alpha^2}{\delta} < r_k \leq \alpha$ . Moreover, we have  $r_0 = \alpha$  and  $\frac{\alpha^2}{\delta} < r_k < \alpha$  for each  $k = 1, 2, \dots$ . For such  $r_k$ , by definition of  $\phi_{\alpha, \delta}(r)$ , we have

$$\phi_{\alpha, \delta}(r_k) = \frac{\delta r_k^2}{\alpha^2}.$$

Thus  $\phi_{\alpha, \delta}^k(r_k) = \alpha$  has the form

$$\phi_{\alpha, \delta}^k(r_k) = \frac{\delta^{2^k-1}}{\alpha^{2(2^k-1)}} r_k^{2^k} = \alpha$$

consequently

$$r_k^{2^k} = \alpha^{2^k} \cdot \left[ \left( \frac{\alpha}{\delta} \right)^{\frac{2^k-1}{2^k}} \right]^{2^k}.$$

Taking  $2^k$ -root we obtain unique positive solution:  $r_k = \alpha \cdot \left( \frac{\alpha}{\delta} \right)^{\frac{2^k-1}{2^k}}$ . □

If  $\alpha = \delta$ , then we denote

$$\hat{\alpha}(x) = |f(x) - x_0|_p, \quad \text{for } x \in S_\alpha(x_0).$$

Using Lemma 3 and Lemma 6 we get

**Theorem 4.** *If  $\alpha = \delta$ , then the  $p$ -adic dynamical system generated by function (3.4) has the following properties:*

- I. I.i)  $SI(x_0) = U_\alpha(x_0)$ .  
 I.ii)  $U_\alpha(x_0) \cap \mathcal{P} = \emptyset$ .
- II. *If  $r > \alpha$  and  $x \in S_r(x_0)$ , then  $f(x) \in S_\alpha(x_0)$ .*
- III. *Let  $f^k(x) \in S_\alpha(x_0) \setminus \mathcal{P}$  for some  $k = 0, 1, 2, \dots$ , then*

$$f^m(x) \in \begin{cases} S_{\hat{\alpha}(f^k(x))}(x_0), & \text{if } \hat{\alpha}(f^k(x)) \geq \alpha, \quad m = k + 1 \\ S_\alpha(x_0), & \text{if } \hat{\alpha}(f^k(x)) > \alpha, \quad m = k + 2 \\ S_{\hat{\alpha}(f^k(x))}(x_0), & \text{if } \hat{\alpha}(f^k(x)) < \alpha, \quad \forall m \geq k + 1. \end{cases}$$

*Proof.* Parts II-III of theorem easily follow from parts II-III of Lemma 3 and Lemma 6.

I. By the part I of Lemma 3 and Lemma 6, if  $r < \alpha$  and  $x \in S_r(x_0)$  then  $|f^n(x) - x_0|_p = \psi^n(r) = r$ , i.e., for  $n \geq 1$  we have  $f^n(x) \in S_r(x_0)$ . Consequently,  $U_\alpha(x_0) \subset SI(x_0)$ .

By the parts II-III of theorem we know that if  $r \geq \alpha$  then  $S_r(x_0)$  is not invariant for  $f$ . Hence  $SI(x_0) \subset U_\alpha(x_0)$ . Therefore,  $SI(x_0) = U_\alpha(x_0)$ .

Since  $|x_0 - x_1|_p = |x_0 - x_2|_p = \alpha$  we have  $x_{1,2} \notin U_\alpha(x_0)$ . Moreover, from  $f(U_\alpha(x_0)) \subset U_\alpha(x_0)$  we get

$$U_\alpha(x_0) \cap \mathcal{P} = \{x \in U_\alpha(x_0) : \exists n \in \mathbb{N} \cup \{0\}, f^n(x) \in \{x_1, x_2\}\} = \emptyset.$$

□

**3.2. Case:  $\alpha < \beta$ .** In this case our arguments are similar to the ones used for the case  $\alpha = \beta$ , therefore we give results of this subsection without proofs.

Consider the following functions:

For  $\delta < \alpha$  define the function  $\varphi_{\alpha,\beta,\delta} : [0, +\infty) \rightarrow [0, +\infty)$  by

$$\varphi_{\alpha,\beta,\delta}(r) = \begin{cases} r, & \text{if } r < \alpha \\ \alpha^*, & \text{if } r = \alpha \\ \alpha, & \text{if } \alpha < r < \beta \\ \beta^*, & \text{if } r = \beta \\ \frac{\alpha\beta}{r}, & \text{if } \beta < r < \frac{\alpha\beta}{\delta} \\ \delta^*, & \text{if } r = \frac{\alpha\beta}{\delta} \\ \delta, & \text{if } r > \frac{\alpha\beta}{\delta} \end{cases}$$

where  $\alpha^*$ ,  $\beta^*$  and  $\delta^*$  some positive numbers with  $\alpha^* \geq \alpha$ ,  $\beta^* \geq \alpha$  and  $\delta^* \leq \delta$ .

For  $\alpha = \delta$  define the function  $\phi_{\alpha,\beta} : [0, +\infty) \rightarrow [0, +\infty)$  by

$$\phi_{\alpha,\beta}(r) = \begin{cases} r, & \text{if } r < \alpha \\ \alpha', & \text{if } r = \alpha \\ \alpha, & \text{if } \alpha < r < \beta \\ \beta', & \text{if } r = \beta \\ \alpha, & \text{if } r > \beta \end{cases}$$

where  $\alpha'$  and  $\beta'$  some positive numbers with  $\alpha' \geq \alpha$ ,  $\beta' > 0$ .

For  $\alpha < \delta < \beta$  define the function  $\psi_{\alpha,\beta,\delta} : [0, +\infty) \rightarrow [0, +\infty)$  by

$$\psi_{\alpha,\beta,\delta}(r) = \begin{cases} r, & \text{if } r < \alpha \\ \hat{\alpha}, & \text{if } r = \alpha \\ \alpha, & \text{if } \alpha < r < \frac{\alpha\beta}{\delta} \\ \hat{\delta}, & \text{if } r = \frac{\alpha\beta}{\delta} \\ \frac{\delta}{\beta}r, & \text{if } \frac{\alpha\beta}{\delta} < r < \beta \\ \hat{\beta}, & \text{if } r = \beta \\ \delta, & \text{if } r > \beta \end{cases}$$

where  $\hat{\alpha}$ ,  $\hat{\beta}$  and  $\hat{\delta}$  some positive numbers with  $\hat{\alpha} \geq \alpha$ ,  $\hat{\beta} \geq \delta$  and  $\hat{\delta} \leq \alpha$ .

For  $\delta = \beta$  define the function  $\eta_{\alpha,\beta} : [0, +\infty) \rightarrow [0, +\infty)$  by

$$\eta_{\alpha,\beta}(r) = \begin{cases} r, & \text{if } r < \alpha \\ \bar{\alpha}, & \text{if } r = \alpha \\ r, & \text{if } \alpha < r < \beta \\ \bar{\beta}, & \text{if } r = \beta \\ \beta, & \text{if } r > \beta \end{cases}$$

where  $\bar{\alpha}$  and  $\bar{\beta}$  some positive numbers with  $\bar{\alpha} > 0$ ,  $\bar{\beta} \geq \beta$ .

For  $\beta < \delta$  define the function  $\zeta_{\alpha,\beta,\delta} : [0, +\infty) \rightarrow [0, +\infty)$  by

$$\zeta_{\alpha,\beta,\delta}(r) = \begin{cases} r, & \text{if } r < \frac{\alpha\beta}{\delta} \\ \tilde{\delta}, & \text{if } r = \frac{\alpha\beta}{\delta} \\ \frac{\delta r^2}{\alpha\beta}, & \text{if } \frac{\alpha\beta}{\delta} < r < \alpha \\ \tilde{\alpha}, & \text{if } r = \alpha \\ \frac{\delta}{\beta}r, & \text{if } \alpha < r < \beta \\ \tilde{\beta}, & \text{if } r = \beta \\ \delta, & \text{if } r > \beta \end{cases}$$

where  $\tilde{\alpha}$ ,  $\tilde{\beta}$  and  $\tilde{\delta}$  some positive numbers with  $\tilde{\alpha} \geq \frac{\alpha\delta}{\beta}$ ,  $\tilde{\beta} \geq \delta$  and  $\tilde{\delta} \leq \frac{\alpha\beta}{\delta}$ .

Using the formula (3.5) we easily get the following:

**Lemma 7.** *If  $\alpha < \beta$  and  $x \in S_r(x_0) \setminus \mathcal{P}$ , then the following formula holds for function (3.4)*

$$|f^n(x) - x_0|_p = \begin{cases} \varphi_{\alpha,\beta,\delta}^n(r), & \text{if } \delta < \alpha \\ \phi_{\alpha,\beta}^n(r), & \text{if } \alpha = \delta \\ \psi_{\alpha,\beta,\delta}^n(r), & \text{if } \alpha < \delta < \beta \\ \eta_{\alpha,\beta}^n(r), & \text{if } \delta = \beta \\ \zeta_{\alpha,\beta,\delta}^n(r), & \text{if } \beta < \delta \end{cases}$$

Thus the  $p$ -adic dynamical system  $f^n(x), n \geq 1, x \in \mathbb{C}_p \setminus \mathcal{P}$  is related to the real dynamical systems generated by  $\varphi_{\alpha,\beta,\delta}$ ,  $\phi_{\alpha,\beta}$ ,  $\psi_{\alpha,\beta,\delta}$ ,  $\eta_{\alpha,\beta}$  and  $\zeta_{\alpha,\beta,\delta}$ .

The following simple lemmas are devoted to properties of these (real) dynamical systems.

**Lemma 8.** *If  $\delta < \alpha$ , then the dynamical system generated by  $\varphi_{\alpha,\beta,\delta}(r)$  has the following properties:*

1.  $\text{Fix}(\varphi_{\alpha,\beta,\delta}) = \{r : 0 \leq r < \alpha\} \cup \{\alpha : \text{if } \alpha^* = \alpha\} \cup \{\beta : \text{if } \beta^* = \beta\}$ .
2. *If  $\alpha < r < \beta$ , then  $\varphi_{\alpha,\beta,\delta}(r) = \alpha$ .*
3. *If  $r > \beta$ , then*

$$\varphi_{\alpha,\beta,\delta}^n(r) = \begin{cases} \frac{\alpha\beta}{r}, & \text{for all } \beta < r < \frac{\alpha\beta}{\delta} \\ \delta^*, & \text{for } r = \frac{\alpha\beta}{\delta} \\ \delta, & \text{for all } r > \frac{\alpha\beta}{\delta} \end{cases}$$

*for any  $n \geq 1$ .*

4. *Let  $r = \alpha$ .*
  - 4.1) *If  $\alpha < \alpha^* < \beta$ , then  $\varphi_{\alpha,\beta,\delta}^2(\alpha) = \alpha$ .*
  - 4.2) *If  $\alpha^* = \beta$ , then  $\varphi_{\alpha,\beta,\delta}(\alpha) = \beta$ .*

4.3) If  $\alpha^* > \beta$ , then

$$\varphi_{\alpha,\beta,\delta}^n(\alpha) = \begin{cases} \frac{\alpha\beta}{\alpha^*}, & \text{if } \beta < \alpha^* < \frac{\alpha\beta}{\delta} \\ \delta^*, & \text{if } \alpha^* = \frac{\alpha\beta}{\delta} \\ \delta, & \text{if } \alpha^* > \frac{\alpha\beta}{\delta} \end{cases}$$

for any  $n \geq 2$ .

5. Let  $r = \beta$ .

5.1) If  $\alpha < \beta^* < \beta$ , then  $\varphi_{\alpha,\beta,\delta}^2(\beta) = \alpha$ .

5.2) If  $\beta^* > \beta$ , then

$$\varphi_{\alpha,\beta,\delta}^n(\beta) = \begin{cases} \frac{\alpha\beta}{\beta^*}, & \text{if } \beta < \beta^* < \frac{\alpha\beta}{\delta} \\ \delta^*, & \text{if } \beta^* = \frac{\alpha\beta}{\delta} \\ \delta, & \text{if } \beta^* > \frac{\alpha\beta}{\delta} \end{cases}$$

for any  $n \geq 2$ .

**Lemma 9.** If  $\alpha = \delta$ , then the dynamical system generated by  $\phi_{\alpha,\beta}(r)$  has the following properties:

1.  $\text{Fix}(\phi_{\alpha,\beta}) = \{r : 0 \leq r < \alpha\} \cup \{\alpha : \text{if } \alpha' = \alpha\} \cup \{\beta : \text{if } \beta' = \beta\}$ .
2. If  $r > \alpha$  and  $r \neq \beta$ , then  $\phi_{\alpha,\beta}(r) = \alpha$ .
3. Let  $r = \alpha$  and  $\alpha' > \alpha$ .
  - 3.1) If  $\alpha' \neq \beta$ , then  $\phi_{\alpha,\beta}^2(\alpha) = \alpha$ .
  - 3.2) If  $\alpha' = \beta$ , then  $\phi_{\alpha,\beta}(\alpha) = \beta$ .
4. If  $r = \beta$ 
  - 4.1) If  $\beta' < \alpha$ , then  $\phi_{\alpha,\beta}^n(\beta) = \beta'$  for all  $n \geq 1$ .
  - 4.2) If  $\beta' = \alpha$ , then  $\phi_{\alpha,\beta}(\beta) = \alpha$ .
  - 4.3) If  $\beta' > \alpha$  and  $\beta' \neq \beta$ , then  $\phi_{\alpha,\beta}^2(\beta) = \alpha$ .

**Lemma 10.** If  $\alpha < \delta < \beta$ , then the dynamical system generated by  $\psi_{\alpha,\beta,\delta}(r)$  has the following properties:

1.  $\text{Fix}(\psi_{\alpha,\beta,\delta}) = \{r : 0 \leq r < \alpha\} \cup \{\alpha : \text{if } \hat{\alpha} = \alpha\} \cup \{\beta : \text{if } \hat{\beta} = \beta\}$ .
2. If  $\hat{\alpha} \notin \{\alpha, \beta\}$  and  $\hat{\beta} \neq \beta$ , then there exists  $n \in \mathbb{N}$  such that  $\psi_{\alpha,\beta,\delta}^n(r) = \alpha$  for all  $r \geq \alpha$ .
3. If  $r = \alpha$  and  $\hat{\alpha} = \beta$ , then  $\psi_{\alpha,\beta,\delta}(\alpha) = \beta$ .

**Lemma 11.** If  $\delta = \beta$ , then the dynamical system generated by  $\eta_{\alpha,\beta}(r)$  has the following properties:

1.  $\text{Fix}(\eta_{\alpha,\beta}) = \{r : 0 \leq r < \alpha\} \cup \{\alpha : \text{if } \bar{\alpha} = \alpha\} \cup \{r : \alpha < r < \beta\} \cup \{\beta : \text{if } \bar{\beta} = \beta\}$ .
2. If  $r > \beta$ , then  $\eta_{\alpha,\beta}(r) = \beta$ .
3. Let  $r = \alpha$ .
  - 3.1) If  $\bar{\alpha} \neq \alpha$  and  $\bar{\alpha} < \beta$ , then  $\eta_{\alpha,\beta}^n(\alpha) = \bar{\alpha}$ , for all  $n \geq 1$ .
  - 3.2) If  $\bar{\alpha} = \beta$ , then  $\eta_{\alpha,\beta}(\alpha) = \beta$ .
  - 3.3) If  $\bar{\alpha} > \beta$ , then  $\eta_{\alpha,\beta}^2(\alpha) = \beta$ .



4. If  $r = \beta$  and  $\bar{\beta} > \beta$ , then  $\eta_{\alpha,\beta}^2(\beta) = \beta$ .

**Lemma 12.** *If  $\delta > \beta$ , then the dynamical system generated by  $\zeta_{\alpha,\beta,\delta}(r)$  has the following properties:*

1.  $\text{Fix}(\zeta_{\alpha,\beta,\delta}) = \{r : 0 \leq r < \frac{\alpha\beta}{\delta}\} \cup \{\frac{\alpha\beta}{\delta} : \text{if } \tilde{\delta} = \frac{\alpha\beta}{\delta}\} \cup \{\delta\}$ .
2. If  $r > \frac{\alpha\beta}{\delta}$ , then

$$\lim_{n \rightarrow \infty} \zeta_{\alpha,\beta,\delta}^n(r) = \delta.$$

3. If  $r = \frac{\alpha\beta}{\delta}$  and  $\tilde{\delta} < \frac{\alpha\beta}{\delta}$ , then

$$\zeta_{\alpha,\beta,\delta}^n(r) = \tilde{\delta}$$

for any  $n \geq 1$ .

If  $\alpha < \beta$ , then we note that  $\mathcal{P}$  has the following form  $\mathcal{P} = \mathcal{P}_\alpha \cup \mathcal{P}_\beta$ , where

$$\mathcal{P}_\alpha = \{x \in \mathbb{C}_p : \exists n \in \mathbb{N} \cup \{0\}, f^n(x) = x_1\} \quad \text{and} \quad \mathcal{P}_\beta = \{x \in \mathbb{C}_p : \exists n \in \mathbb{N} \cup \{0\}, f^n(x) = x_2\}.$$

By definitions of  $\varphi_{\alpha,\beta,\delta}$ ,  $\phi_{\alpha,\beta}$ ,  $\psi_{\alpha,\beta,\delta}$ ,  $\eta_{\alpha,\beta}$  and  $\zeta_{\alpha,\beta,\delta}$  and Lemma 8 - Lemma 12 we have the following

**Theorem 5.** *If  $\alpha < \beta$ , then*

1. If  $\delta < \alpha$ , then

$$\mathcal{P}_\alpha \subset \bigcup_{\alpha \leq r \leq \beta} S_r(x_0) \quad \text{and} \quad \mathcal{P}_\beta \subset S_\beta(x_0).$$

2. If  $\alpha \leq \delta < \beta$ , then

$$\mathcal{P}_\alpha \subset \bigcup_{r \geq \alpha} S_r(x_0) \quad \text{and} \quad \mathcal{P}_\beta \subset S_\beta(x_0).$$

3. If  $\delta = \beta$ , then

$$\mathcal{P}_\alpha \subset S_\alpha(x_0) \quad \text{and} \quad \mathcal{P}_\beta \subset \bigcup_{r \geq \beta} S_r(x_0).$$

4. If  $\delta > \beta$ , then there are two sequences  $\{r_k\} \subset (\frac{\alpha\beta}{\delta}, \alpha]$  and  $\{\rho_k\} \subset (\frac{\alpha\beta}{\delta}, \beta]$  such that

$$\mathcal{P}_\alpha \subset \bigcup_{k=1}^{\infty} S_{r_k}(x_0) \quad \text{and} \quad \mathcal{P}_\beta \subset \bigcup_{k=1}^{\infty} S_{\rho_k}(x_0).$$

Now we shall apply above-mentioned results for study of the  $p$ -adic dynamical system generated by function (3.4).

For  $\alpha < \beta$  denote the following

$$\begin{aligned} \alpha^*(x) &= |f(x) - x_0|_p, \quad \text{if } x \in S_\alpha(x_0) \setminus \{x_1\}; \\ \beta^*(x) &= |f(x) - x_0|_p, \quad \text{if } x \in S_\beta(x_0) \setminus \{x_2\}; \\ \delta^*(x) &= |f(x) - x_0|_p, \quad \text{if } x \in S_{\frac{\alpha\beta}{\delta}}(x_0). \end{aligned}$$

Then using Lemma 7 and Lemma 8 we obtain the following

**Theorem 6.** *If  $\delta < \alpha < \beta$  and  $x \in S_r(x_0) \setminus \mathcal{P}$ , then the  $p$ -adic dynamical system generated by function (3.4) has the following properties:*

1.  $SI(x_0) = U_\alpha(x_0)$ .
2. If  $\alpha < r < \beta$ , then  $f(x) \in S_\alpha(x_0)$ .
3. Let  $r > \beta$ , then

$$f^n(x) \in \begin{cases} S_{\frac{\alpha\beta}{r}}(x_0), & \text{for all } \alpha < r < \frac{\alpha\beta}{\delta} \\ S_{\delta^*(x)}(x_0), & \text{for } r = \frac{\alpha\beta}{\delta} \\ S_\delta(x_0), & \text{for all } r > \frac{\alpha\beta}{\delta}, \end{cases}$$

for any  $n \geq 1$ .

4. Let  $x \in S_\alpha(x_0) \setminus \mathcal{P}$ .
  - 4.1) If  $\alpha^*(x) = \alpha$ , then  $f(x) \in S_\alpha(x_0)$ .
  - 4.2) If  $\alpha < \alpha^*(x) < \beta$ , then  $f^2(x) \in S_\alpha(x_0)$ .
  - 4.3) If  $\alpha^*(x) = \beta$ , then  $f(x) \in S_\beta(x_0)$ .
  - 4.4) If  $\alpha^*(x) > \beta$ , then

$$f^n(x) \in \begin{cases} S_{\frac{\alpha\beta}{\alpha^*(x)}}(x_0), & \text{for all } \alpha < \alpha^*(x) < \frac{\alpha\beta}{\delta} \\ S_{\delta^*(f(x))}(x_0), & \text{for } \alpha^*(x) = \frac{\alpha\beta}{\delta} \\ S_\delta(x_0), & \text{for all } \alpha^*(x) > \frac{\alpha\beta}{\delta} \end{cases}$$

for any  $n \geq 2$ .

5. Let  $x \in S_\beta(x_0) \setminus \mathcal{P}$ .
  - 5.1) If  $\beta^*(x) = \alpha$ , then  $f(x) \in S_\alpha(x_0)$ .
  - 5.2) If  $\alpha < \beta^*(x) < \beta$ , then  $f^2(x) \in S_\alpha(x_0)$ .
  - 5.3) If  $\beta^*(x) = \beta$ , then  $f(x) \in S_\beta(x_0)$ .
  - 5.4) If  $\beta^*(x) > \beta$ , then

$$f^n(x) \in \begin{cases} S_{\frac{\alpha\beta}{\beta^*(x)}}(x_0), & \text{for all } \alpha < \beta^*(x) < \frac{\alpha\beta}{\delta} \\ S_{\delta^*(f(x))}(x_0), & \text{for } \beta^*(x) = \frac{\alpha\beta}{\delta} \\ S_\delta(x_0), & \text{for all } \beta^*(x) > \frac{\alpha\beta}{\delta} \end{cases}$$

for any  $n \geq 2$ .

By Lemma 7 and Lemma 9 we get

**Theorem 7.** *If  $\alpha = \delta$  and  $x \in S_r(x_0) \setminus \mathcal{P}$ , then the  $p$ -adic dynamical system generated by function (3.4) has the following properties:*

1.  $SI(x_0) = U_\alpha(x_0)$ .
2. If  $r > \alpha$  and  $r \neq \beta$ , then  $f(x) \in S_\alpha(x_0)$ .
3. Let  $x \in S_\alpha(x_0) \setminus \mathcal{P}$ .
  - 3.1) If  $\alpha^*(x) = \alpha$ , then  $f(x) \in S_\alpha(x_0)$ .
  - 3.2) If  $\alpha^*(x) > \alpha$  and  $\alpha^*(x) \neq \beta$ , then  $f^2(x) \in S_\alpha(x_0)$ .
  - 3.3) If  $\alpha^*(x) = \beta$ , then  $f(x) \in S_\beta(x_0)$ .

4. Let  $x \in S_\beta(x_0) \setminus \mathcal{P}$ .
  - 4.1) If  $\beta^*(x) < \alpha$ , then  $f^n(x) \in S_{\beta^*(x)}(x_0)$  for any  $n \geq 1$ .
  - 4.2) If  $\beta^*(x) = \alpha$ , then  $f(x) \in S_\alpha(x_0)$ .
  - 4.3) If  $\beta^*(x) > \alpha$  and  $\beta^*(x) \neq \beta$ , then  $f^2(x) \in S_\alpha(x_0)$ .
  - 4.4) If  $\beta^*(x) = \beta$ , then  $f(x) \in S_\beta(x_0)$ .

Using the Lemma 7 and Lemma 10 we get

**Theorem 8.** *If  $\alpha < \delta < \beta$  and  $x \in S_r(x_0) \setminus \mathcal{P}$ , then the  $p$ -adic dynamical system generated by function (3.4) has the following properties:*

1.  $SI(x_0) = U_\alpha(x_0)$ .
2. If  $r > \alpha$  and  $r \neq \beta$ , then there exists  $n \in N$  such that  $f^n(x) \in S_\alpha(x_0)$ .
3. Let  $r = \alpha$  ( $r = \beta$ ).
  - 3.1) If  $\alpha^*(x) \neq \beta$  ( $\beta^*(x) \neq \beta$ ), then there exists  $n \in N$  such that  $f^n(x) \in S_\alpha(x_0)$ .
  - 3.2) If  $\alpha^*(x) = \beta$  ( $\beta^*(x) = \beta$ ), then  $f(x) \in S_\beta(x_0)$ .

Using the Lemma 7 and Lemma 11 we get

**Theorem 9.** *If  $\delta = \beta$  and  $x \in S_r(x_0) \setminus \mathcal{P}$ , then the  $p$ -adic dynamical system generated by function (3.4) has the following properties:*

1. 1.1)  $SI(x_0) = U_\alpha(x_0)$ .
  - 1.2) If  $\alpha < r < \beta$ , then  $S_r(x_0)$  is an invariant for  $f$ .
2. Let  $r = \alpha$  and  $x \in S_\alpha(x_0) \setminus \mathcal{P}$ .
  - 2.1) If  $\alpha^*(x) = \alpha$ , then  $f(x) \in S_\alpha(x_0)$ .
  - 2.2) If  $\alpha^*(x) < \beta$  and  $\alpha^*(x) \neq \alpha$ , then  $f^n(x) \in S_{\alpha^*(x)}(x_0)$  for any  $n \geq 1$ .
  - 2.3) If  $\alpha^*(x) = \beta$ , then  $f(x) \in S_\beta(x_0)$ .
  - 2.4) If  $\alpha^*(x) > \beta$ , then  $f^2(x) \in S_\beta(x_0)$ .
3. Let  $r = \beta$  and  $x \in S_\beta(x_0) \setminus \mathcal{P}$ .
  - 3.1) If  $\beta^*(x) = \beta$ , then  $f(x) \in S_\beta(x_0)$ .
  - 3.2) If  $\beta^*(x) > \beta$ , then  $f^2(x) \in S_\beta(x_0)$ .
4. If  $r > \beta$ , then  $f(S_r(x_0)) \subset S_\beta(x_0)$ .

By Lemma 7 and Lemma 12 we get

**Theorem 10.** *If  $\delta > \beta$  and  $x \in S_r(x_0) \setminus \mathcal{P}$ , then the  $p$ -adic dynamical system generated by function (3.4) has the following properties:*

1. 1.1)  $SI(x_0) = U_{\frac{\alpha\beta}{\delta}}(x_0)$ .
  - 1.2) The sphere  $S_\delta(x_0)$  is invariant for  $f$ .
2. If  $r > \frac{\alpha\beta}{\delta}$ , then
 
$$\lim_{n \rightarrow \infty} f^n(x) \in S_\delta(x_0).$$
3. If  $r = \frac{\alpha\beta}{\delta}$ , then one of the following two possibilities holds:
  - 3.1) There exists  $k \in N$  and  $\mu_k < \frac{\alpha\beta}{\delta}$  such that
 
$$f^m(x) \in S_{\mu_k}(x_0)$$
 for any  $m \geq k$  and  $f^m(x) \in S_{\frac{\alpha\beta}{\delta}}(x_0)$  if  $m \leq k - 1$ .

3.2) The trajectory  $\{f^k(x), k \geq 1\}$  is a subset of  $S_{\frac{\alpha\beta}{\delta}}(x_0)$ .

**Remark 3.** If  $\alpha < \beta$  and  $p \geq 3$  then by Lemma 2 we have only Theorem 9 and Theorem 10. See Remark 2 for the case  $\alpha > \beta$ .

#### 4. DYNAMICAL SYSTEM $f(x) = \frac{ax^2+bx}{x^2+ax+b}$ IN $Q_p$ IS NOT ERGODIC

In this section we assume  $a = d$  in (3.4), and suppose that the square root  $\sqrt{a^2 - 4b}$  exists in  $Q_p$ . Then (3.4) has the form

$$f(x) = \frac{ax^2 + bx}{x^2 + ax + b} \quad (4.1)$$

where  $x \neq \hat{x}_{1,2} = \frac{-a \pm \sqrt{a^2 - 4b}}{2}$ .

Consider the dynamical system (4.1) in  $Q_p$ . It is easy to see that  $x_0 = 0$  is unique fixed point for (4.1).

We have  $\delta = |a|_p$ ,  $\alpha = |\hat{x}_1|_p$ ,  $\beta = |\hat{x}_2|_p$  and  $|b|_p = \alpha\beta$ .

Since  $\hat{x}_1 + \hat{x}_2 = -a$  we have  $\delta \leq \max\{\alpha, \beta\}$ .

Define the following sets

$$A_1 = \{r : 0 \leq r < \alpha\} \text{ if } \delta \leq \alpha = \beta;$$

$$A_2 = \{r : r \in [0, \beta) \setminus \{\alpha\}\} \text{ if } \alpha < \beta = \delta;$$

and we denote  $A = A_1 \cup A_2$  for  $\alpha \leq \beta$ .

From previous sections we have the following

**Corollary 1.** The sphere  $S_r(0)$  is invariant for  $f$  if and only if  $r \in A$ .

Since  $x_0 = 0$  is an indifferent fixed point, in this section we are interested to study ergodicity properties of the dynamical system.

**Lemma 13.** For every closed ball  $V_\rho(c) \subset S_r(0)$ ,  $r \in A$  the following equality holds

$$f(V_\rho(c)) = V_\rho(f(c)).$$

*Proof.* From inclusion  $V_\rho(c) \subset S_r(0)$  we have  $|c|_p = r$ .

Let  $x \in V_\rho(c)$ , i.e.  $|x - c|_p \leq \rho$ , then

$$|f(x) - f(c)|_p = |x - c|_p \cdot \frac{|a^2cx + ab(x+c) - bcx + b^2|_p}{|(x - \hat{x}_1)(x - \hat{x}_2)(c - \hat{x}_1)(c - \hat{x}_2)|_p}. \quad (4.2)$$

We have

$$|a^2cx|_p = \delta^2 r^2, \quad |ab(x+c)|_p \leq \alpha\beta\delta r, \quad |bcx|_p = \alpha\beta r^2, \quad |b^2|_p = \alpha^2\beta^2.$$

If  $r \in A_1$ , then  $\max\{\delta^2 r^2, \alpha\beta\delta r, \alpha\beta r^2, \alpha^2\beta^2\} = \alpha^2\beta^2$  and

$$|(x - \hat{x}_1)(x - \hat{x}_2)(c - \hat{x}_1)(c - \hat{x}_2)|_p = \alpha^2\beta^2.$$

Using this equality by (4.2) we get  $|f(x) - f(c)|_p = |x - c|_p \leq \rho$ .

If  $\alpha < \beta = \delta$ , then  $r \in A_2$ . Consequently,  $r \in [0, \alpha)$  or  $r \in (\alpha, \beta)$ .

Let  $0 \leq r < \alpha$ . Then  $|f(x) - f(c)|_p = |x - c|_p \leq \rho$ .

Let  $\alpha < r < \beta$ . Then  $\max\{\delta^2 r^2, \alpha\beta\delta r, \alpha\beta r^2, \alpha^2\beta^2\} = \delta^2 r^2$  and

$$|(x - \hat{x}_1)(x - \hat{x}_2)(c - \hat{x}_1)(c - \hat{x}_2)|_p = r^2\beta^2.$$

Consequently  $|f(x) - f(c)|_p = |x - c|_p \leq \rho$ . This completes the proof.  $\square$

Recall that  $S_r(0)$  is invariant with respect to  $f$  iff  $r \in A$ .

**Lemma 14.** *If  $c \in S_r(0)$ , where  $r \in A$ , then*

$$|f(c) - c|_p = \begin{cases} \frac{r^3}{\alpha\beta}, & \text{if } r < \alpha \\ \frac{r^2}{\beta}, & \text{if } \alpha < r < \beta \end{cases}$$

*Proof.* Follows from the following equality

$$|f(c) - c|_p = \left| \frac{-c^3}{(c - \hat{x}_1)(c - \hat{x}_2)} \right|_p = \begin{cases} \frac{r^3}{\alpha\beta}, & \text{if } r < \alpha \\ \frac{r^2}{\beta}, & \text{if } \alpha < r < \beta. \end{cases}$$

$\square$

By Lemma 14 we have that  $|f(c) - c|_p$  depends on  $r$ , but does not depend on  $c \in S_r(0)$  itself, therefore we define  $\rho(r) = |f(c) - c|_p$ , if  $c \in S_r(0)$ .

**Theorem 11.** *If  $c \in S_r(0)$ ,  $r \in A$  then*

1. *For any  $n \geq 1$  the following equality holds*

$$|f^{n+1}(c) - f^n(c)|_p = \rho(r). \quad (4.3)$$

2.  *$f(V_{\rho(r)}(c)) = V_{\rho(r)}(c)$ .*

3. *If for some  $\theta > 0$  the ball  $V_\theta(c) \subset S_r(0)$  is an invariant for  $f$ , then*

$$\theta \geq \rho(r).$$

*Proof.* 1. Since  $S_r(0)$  is an invariant of  $f$ , for any  $c \in S_r(0)$  and  $n \geq 1$  we have  $f^n(c) \in S_r(0)$ . Take  $x = f(c)$  then by (4.2) and proof of Lemma 13 we get  $|f(x) - f(c)|_p = |x - c|_p$ , consequently  $|f^2(c) - f(c)|_p = |f(c) - c|_p = \rho(r)$ . Thus for  $n = 1$  the equality (4.3) is true. We use the method of mathematical induction. Assume the equality (4.3) is true for  $n = k$ , i.e.,

$$|f^{k+1}(c) - f^k(c)|_p = |f(c) - c|_p = \rho(r).$$

We shall prove it for  $n = k + 1$ . Denote  $x_1 = f^{k+1}(c)$  and  $c_1 = f^k(c)$  then by (4.2) and proof of Lemma 13 we get  $|f(x_1) - f(c_1)|_p = |x_1 - c_1|_p$ . Therefore

$$|f(x_1) - f(c_1)|_p = |f^{k+2}(c) - f^{k+1}(c)|_p = |f^{k+1}(c) - f^k(c)|_p = |f(c) - c|_p = \rho(r).$$

This completes the proof.

2. Since  $|f(c) - c|_p = \rho(r)$ , we have  $f(c) \in V_{\rho(r)}(c)$  and  $c \in V_{\rho(r)}(f(c))$ . By Lemma 13 and the fact that any point of a ball is its center we get

$$f(V_{\rho(r)}(c)) = V_{\rho(r)}(f(c)) = V_{\rho(r)}(c).$$

3. Let  $V_\theta(c) \subset S_r(0)$  and the ball  $V_\theta(c)$  is an invariant for  $f$ , then  $f(c) \in V_\theta(c)$ , i.e.  $|f(c) - c|_p \leq \theta$ . We have  $\rho(r) = |f(c) - c|_p$  for every  $c \in S_r(0)$ . So then  $\rho(r) \leq \theta$ .  $\square$

For each  $r \in A$  consider a measurable space  $(S_r(0), \mathcal{B})$ , here  $\mathcal{B}$  is the algebra generated by closed subsets of  $S_r(0)$ . Every element of  $\mathcal{B}$  is a union of some balls  $V_\rho(c)$ .

A measure  $\bar{\mu} : \mathcal{B} \rightarrow \mathbb{R}$  is said to be *Haar measure* if it is defined by  $\bar{\mu}(V_\rho(c)) = \rho$ . We consider normalized Haar measure:

$$\mu(V_\rho(c)) = \frac{\bar{\mu}(V_\rho(c))}{\bar{\mu}(S_r(0))} = \frac{\rho}{r}, \quad r > 0.$$

By Lemma 13 we conclude that  $f$  preserves the measure  $\mu$ , i.e.

$$\mu(f(V_\rho(c))) = \mu(V_\rho(c)). \quad (4.4)$$

Consider the dynamical system  $(X, T, \mu)$ , where  $T : X \rightarrow X$  is a measure preserving transformation, and  $\mu$  is a measure. We say that the dynamical system is *ergodic* (or alternatively that  $T$  is ergodic with respect to  $\mu$  or that  $\mu$  is ergodic with respect to  $T$ ) if for every invariant set  $V$  we have  $\mu(V) = 0$  or  $\mu(V) = 1$  (see [23]).

**Theorem 12.** *The dynamical system  $(S_r(0), f, \mu)$  is not ergodic for any  $r \in A$ ,  $r > 0$  (i.e. for any invariant sphere  $S_r(0)$ ). Here  $\mu$  is the normalized Haar measure.*

*Proof.* If a sphere  $S_r(0)$  is invariant for  $f$ , then  $0 \leq r < \alpha$  or  $\alpha < r < \beta$ . By the part 2 of Theorem 11, the ball  $V_{\rho(r)}(c)$  is invariant for any  $c \in S_r(0)$ . Using Lemma 14 we get

$$\mu(V_{\rho(r)}(c)) = \frac{\rho(r)}{r} = \begin{cases} \frac{r^2}{\alpha\beta}, & \text{if } 0 \leq r < \alpha \\ \frac{r}{\beta}, & \text{if } \alpha < r < \beta \end{cases}$$

Since  $0 < r < \alpha \leq \beta$  we have  $0 < \frac{r^2}{\alpha\beta} < 1$ . If  $\alpha < r < \beta$  then  $0 < \frac{r}{\beta} < 1$ . Therefore the dynamical system  $(S_r(0), f, \mu)$  is not ergodic for all  $r \in A$ . □

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U. A. ROZIKOV, INSTITUTE OF MATHEMATICS, 29, DO'RMON YO'LI STR., 100125, TASHKENT, UZBEK-ISTAN.

*E-mail address:* rozikovu@yandex.ru

I. A. SATTAROV, INSTITUTE OF MATHEMATICS, 29, DO'RMON YO'LI STR., 100125, TASHKENT, UZBEK-ISTAN.

*E-mail address:* iskandar1207@rambler.ru